Program : **B.Tech**

Subject Name: **Structural Analysis**

Subject Code: **CE-403**

Semester: **4th**

Unit-4
Arches and Suspension Cables:

Three hinged arches of different shapes, Eddy's Theorem, Suspension cable, stiffening girders, Two Hinged and Fixed Arches - Rib shortening and temperature effects

Three Hinged Arch

Introduction In case of beam support in uniformly distributed load, the maximum bending moment increases with the square of the span and hence they become uneconomical for long span structures. In such situations arches could be advantageously employed, as they would develop horizontal reactions, which in turn reduce the design bending moment.

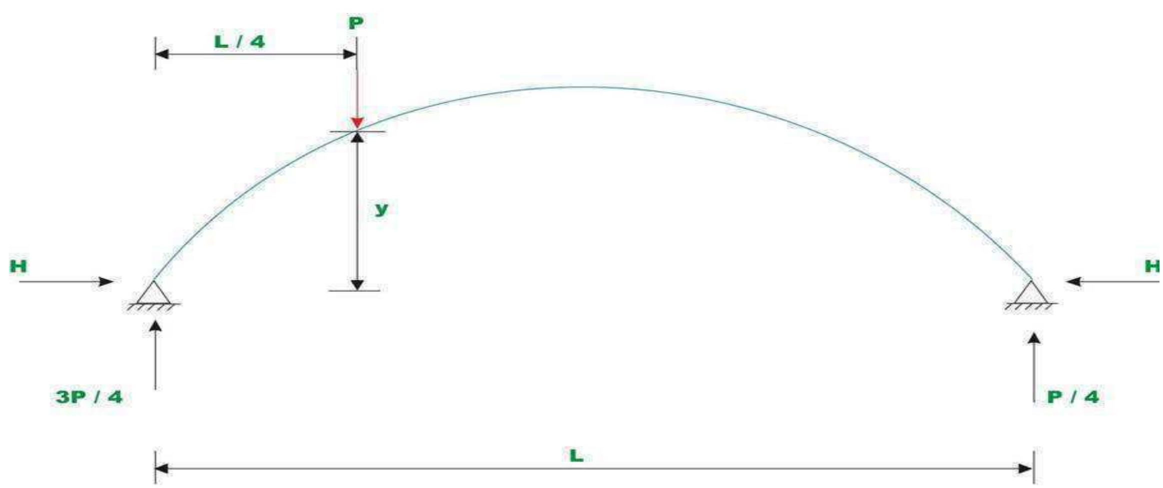


Fig. 32.1 Beam and Arch comparison.

For example,

In the case of a simply supported beam shown in Fig.32.1, the bending moment below the load is 3PL/16. Now consider a two hinged symmetrical arch of the same span and subjected to similar loading as that of simply supported beam. The vertical reaction could be calculated by equations of statics. The horizontal reaction is determined by the method of least work. Now the bending moment below the load is (3PL/16)Hy. It Is clear that the bending moment below the load is reduced in the case of an arch as compared to a simply supported beam. It is observed in the last lesson that, the cable takes the shape of the loading and this shape is termed as funicular shape. If an arch were constructed in an inverted funicular shape then it would be subjected to only compression for those loadings for which its shape is inverted funicular.

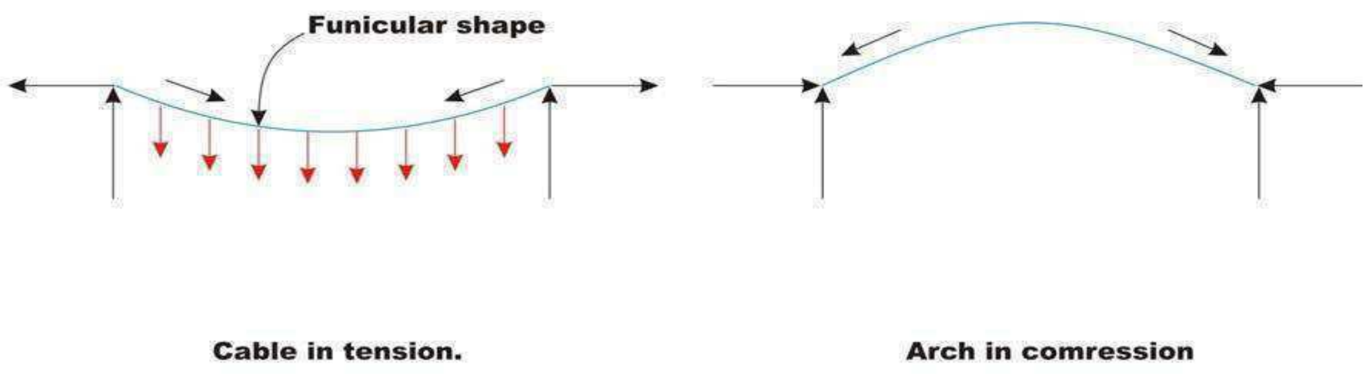


Fig. 32.2 Cable and Arch structure.

Since in practice, the actual shape of the arch differs from the inverted funicular shape or the loading differs from the one for which the arch is an inverted funicular, arches are also subjected to bending moment in addition to compression. As arches are subjected to compression, it must be designed to resist buckling.

Until the beginning of the 20th century, arches and vaults were commonly used to span between walls, piers or other supports. Now, arches are mainly used in bridge construction and do or ways. In earlier days arches were constructed using stones and bricks. In modern times they are being constructed of reinforced concrete and steel.

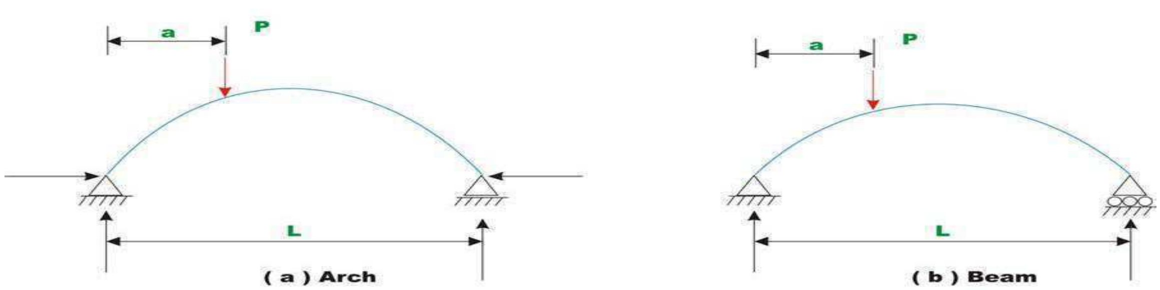
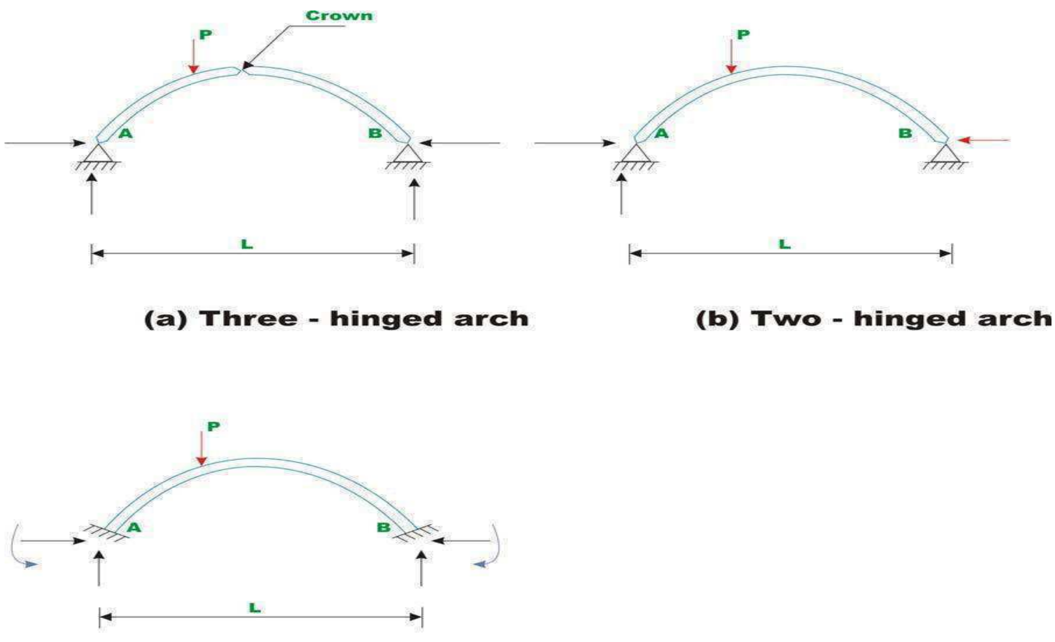


Fig. 32.3



(a) Three - hinged arch (b) Two - hinged arch

(c) Fixed hinged arch

Fig. 32.4 Types of arches.

Analysis of three- hinged arch

In the case of three-hinged arch, we have three hinges: two at the support and one at the crown thus making it statically determinate structure. Consider at three hinged arch subjected to a concentrated force *P as* shown in Fig 32.5.

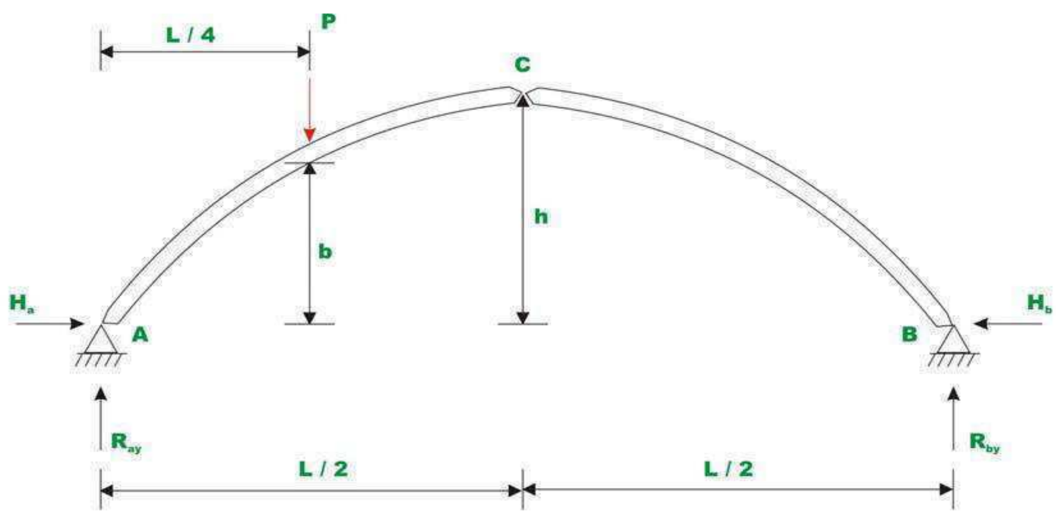


Fig. 32.5 Three hinged arch.

There are four reaction components in the three-hinged arch. One more equation is required in addition to three equations of static equilibrium force valuating the four reaction components. Taking moment about the hinge of all the forces acting on either side of the hinge can set up the

required equation.
Ha = Hb = PL/8h
Va + Vb = Total downwards loads

Example 32.1

A three-hinged parabolic arch of uniform cross section has a span of 60 m and a rise of 10 m. It is subjected to uniformly distributed load of intensity 10 kN/m as shown in Fig. 32.6 Show that the bending moment is zero at any cross section of the arch.

Reactions:
Taking moment of all the forces about hinge *A*, yields
Va = Vb = 10*60/2 = 300 kN.

Taking moment about left hinge c, we get
Va*30 – 10*30*15 – Ha*10 = 0
Ha = Hb = 450 kN.

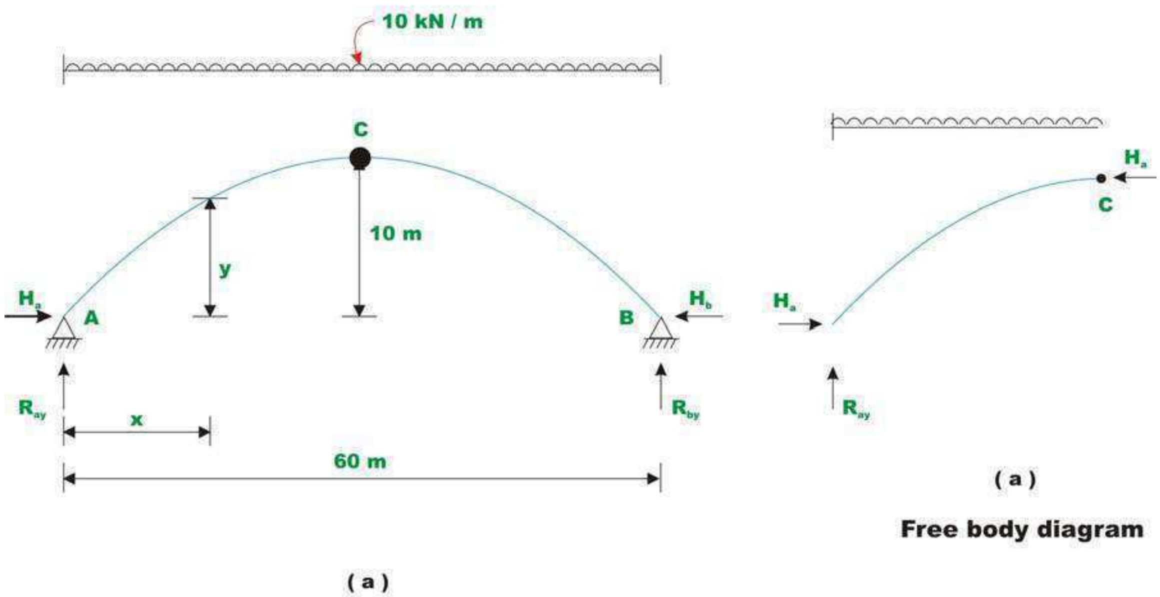


Fig. 32.6 Three hinged arch of Example 32.1

BM at any section XX is given as
BM xx = Va*x – Ha*y – 10*x*(x/2)

The equation of the three-hinged parabolic arch is $y = 4hx(L-x)/L^2$ substituting
BM xx = $300*x – 450(4*10*x*(60-x))\ -5x^2 = 300x – (450(2400x – 40x^2))/(60*60) – 5x^2$
$= 300x – 300x + 5x^2 – 5x^2 = 0$

In other words a three hinged parabolic arch subjected to uniformly distributed load is not subjected to bending moment at any cross section. It supports the load in pure compression.

Example 32.2
A three-hinged semicircular arch of uniform cross section is loaded as shown in Fig 32.7. Calculate the location and magnitude of maximum bending moment in the arch.

Solution:

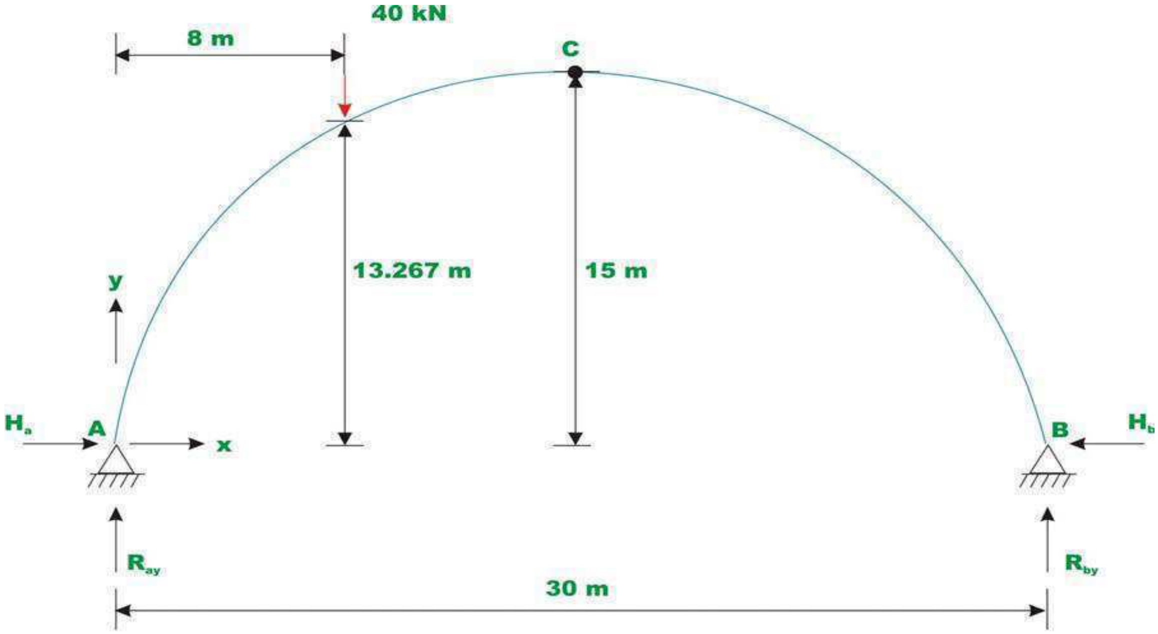


Fig. 32.7 A semi circular arch of Example 32.2

Reactions:
Taking moment of all the forces about hinge *B* leads to,
Va = 29.33kN Vb = 10.67 kN.

Bending moment
Now making use of the condition that the moment at hinge of all the forces left of hinge is zero gives, *CC*
Ha = Hb = 10.66 kN.

The maximum positive bending moment occurs below D and it can be calculated by taking moment of all forces left of about D,
Y at D = $4*15*8(30-8)/30^2$ = 13.267 m.
BM at D = Va*8 – Ha*y = 93.213 kN M.

Example 32.3
A three-hinged parabolic arch is loaded as shown in Fig 32.8a. Calculate the location and magnitude of maximum bending moment in the arch. Draw bending moment diagram.

Solution:

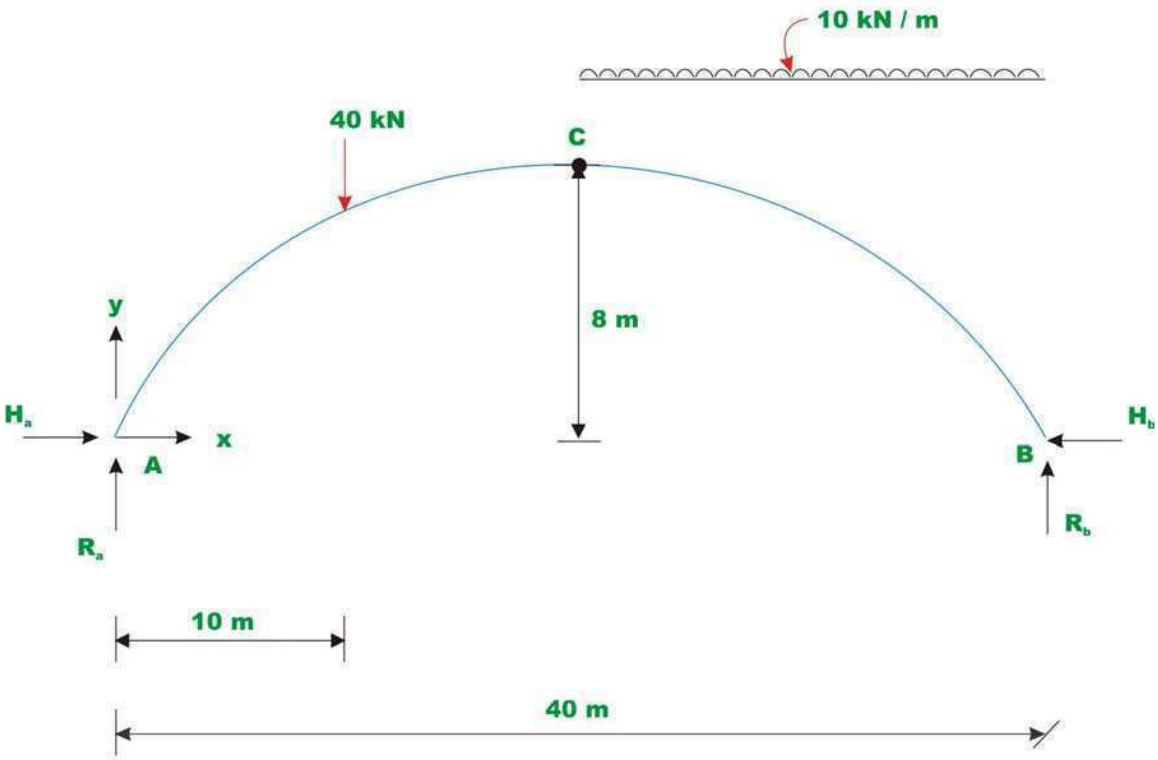


Fig. 32.8a Eaxmple 32.3

Reactions:

Taking *A* as the origin, the equation of the three-hinged parabolic arch is given by,
Va = 80.0 kN Vb = 160.0 kN.
Ha = Hb = 150.0 kN.

Location of maximum bending moment
Consider a section *x* from end *B*. Moment at section *x* in part *CB* of the arch is given by (please note that *B* has been taken as the origin for this calculation),
$BM = 160x – 150y – 10x^2/2$

According to calculus, the necessary condition for extreme (maximum or minimum) is that d(BM)/dx = 0 solving we get x = 10.0 m.
BM max = 200 kN m.

Shear force at D just left of 40 kN load

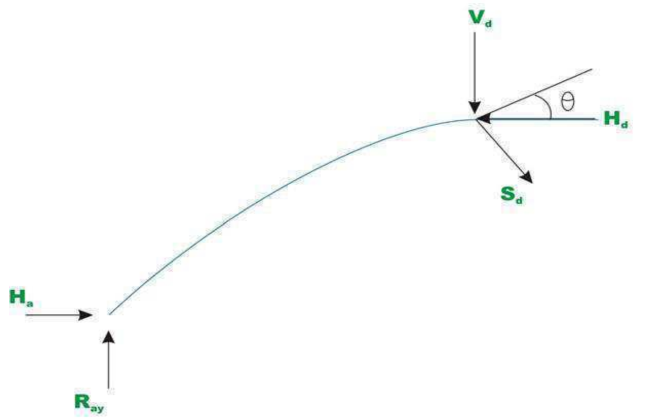


Fig. 32.8b

The slope of the arch at D is evaluated by, tan θ = dy/dx = (8/10) – (16/400)x
Put x = 10.0 m and solving θ = 21.8°

Shear force at left of D **is = Ha Sin θ – Va Cos θ = - 18.57 kN.**

Example 32.4

A three-hinged parabolic arch of constant cross section is subjected to a uniformly distributed load over a part of its span and a concentrated load of 50 kN, as shown in Fig. 32.9. The dimensions of the arch are shown in the figure. Evaluate the horizontal thrust and the maximum bending moment in the arch.

Solution:

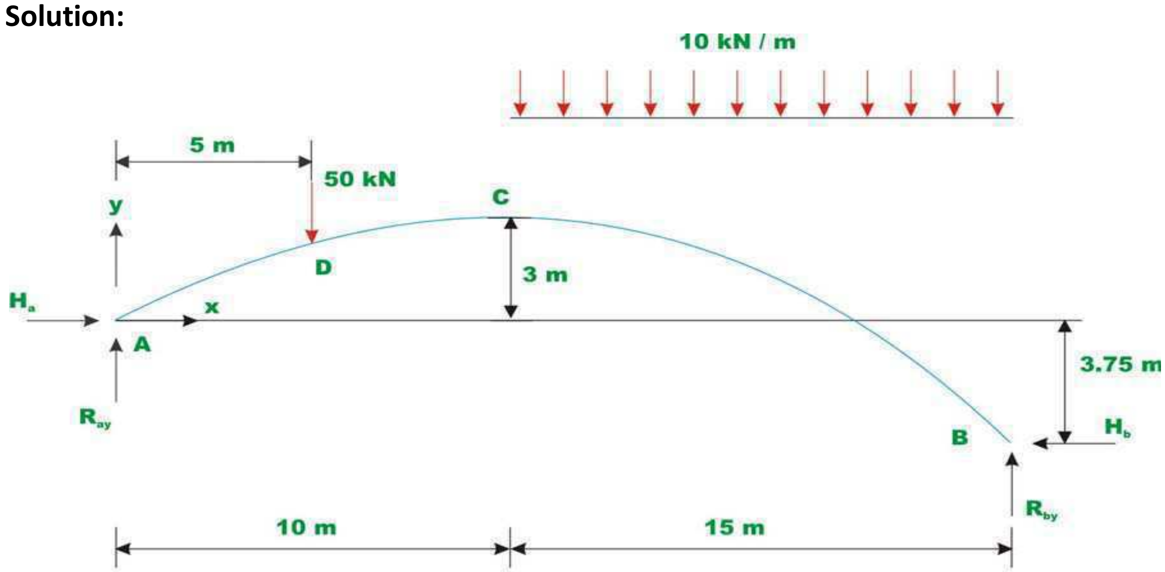


Fig. 32.9

Reactions:

Taking *A* as the origin, the equation of the parabolic arch may be written as,
$Y = 4hx(L-x)/L^2 = 4*3*x(20-x)/20^2 = -0.03x^2 + 0.6x$
Taking moment of all the loads about *B* leads to,
Va*25 + Ha*3.75 - 50*20 – 10*15*7.5 = 0
Va = (2125 – 3.75Ha)/25

Taking moment of all the forces right of hinge *C* about the hinge *C* and setting leads to,
Vb = (1125 + 6.75 Hb)/15
Ha = Hb
Va + Vb = 50 + 10*15 = 200 kN.
Substituting and solving Ha = Hb = 133.33 kN.
Va = 65.0 kN, Vb = 135.0 kN.

Bending moment
From inspection, the maximum negative bending moment occurs in the region *AD* and the maximum positive bending moment occurs in the region*CB*.
Span AD

Bending moment at any cross section in the span AD is
$BM = Va*x – Ha*y = 65*x – 133.33 (-0.03x^2 + 0.6x)$, x lies between 0 to 5
For, the maximum negative bending moment in this region, dBM/dx = 0
Solving x = 1.8748 m.
BM = - 14.06 kN M.

For the maximum positive bending moment in this region occurs at D,
BM = Vb*5 – Hb*y = 135*5 – 133.33 *(-0.03 * 5*5 + 0.6*5) = 25.0 kN M.
Span CB
Bending moment at any cross section, in this span is calculated by,
$BM = Va*x – Ha*(-0.03x^2 + 0.6x) – 50(x – 5) – 10(x – 10) (x – 5)/2$
For locating the position of maximum bending moment, dBM/dx = 0
Solving x = 17.5 m.
BM = 56.25 kN M.
Hence, the maximum positive bending moment occurs in span CB.
Summary

In this lesson, the arch definition is given. The advantages of arch construction are given in the introduction. Arches are classified as three-hinged, two-hinged and hinge less arches. The analysis of three-hinged arch is considered here. Numerical examples are solved in detail to show the general procedure of three-hinged arch analysis.

Two-Hinged Arch

Introduction
Mainly three types of arches are used in practice: three-hinged, two-hinged and hinge less arches. In the early part of the nineteenth century, three-hinged arches were commonly used for

the long span structures as the analysis of such arches could be done with confidence. However, with the development in structural analysis, for long span structures starting from late nineteenth century engineers adopted two-hinged and hinge less arches.

Two-hinged arch is the statically indeterminate structure to degree one. Usually, the horizontal reaction is treated as the redundant and is evaluated by the method of least work. In this lesson, the analysis of two-hinged arches is discussed and few problems are solved to illustrate the procedure for calculating the internal forces.

Analysis of two-hinged arch

A typical two-hinged arch is shown in Fig. 33.1a. In the case of two-hinged arch, we have four unknown reactions, but there are only three equations of equilibrium available. Hence, the degree of statically indeterminacy is one for two-hinged arch.

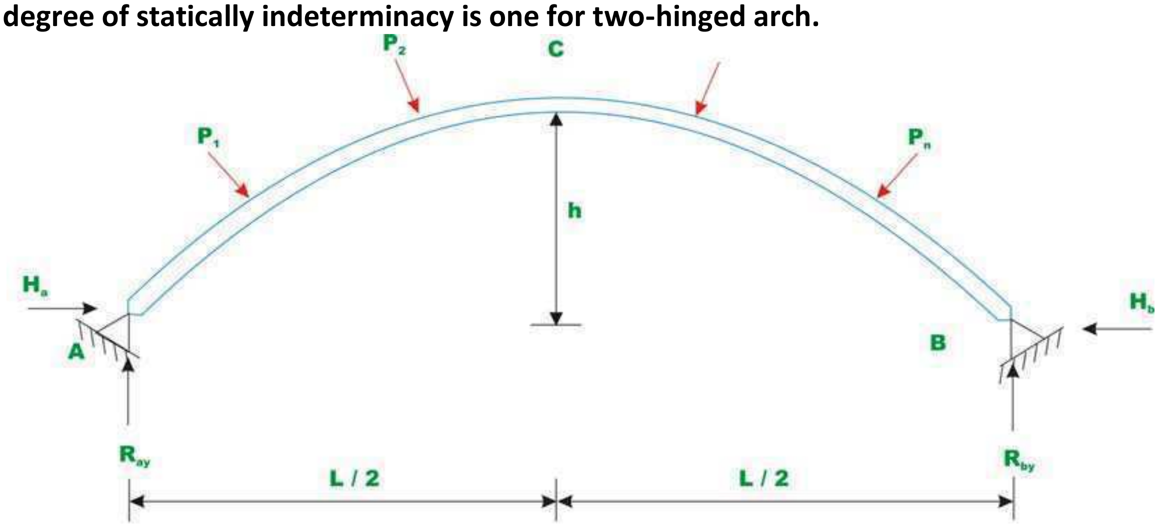


Fig. 33.1a Two - hinged arch.

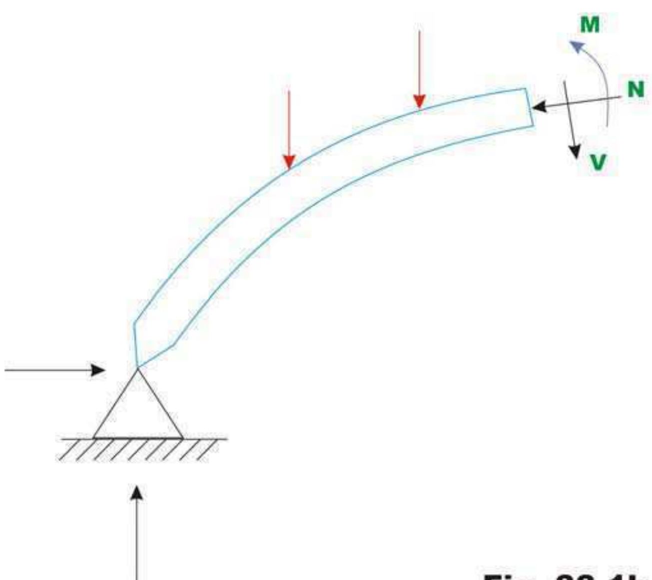


Fig. 33.1b

The fourth equation is written considering deformation of the arch. The unknown redundant reaction Hb is calculated by noting that the horizontal displacement of hinge *b HB* is zero. In general the horizontal reaction in the two hinged arch is evaluated by straightforward application of the theorem of least work (see module 1, lesson 4), which states that the partial derivative of the strain energy of a statically indeterminate structure with respect to statically indeterminate action should vanish.

Hence to obtain, horizontal reaction, one must develop an expression for strain energy. Typically, any section of the arch (vide Fig 33.1b) is subjected to shear force *V*, bending moment *M* and the axial compression. The strain energy Ub due to bending is calculated from the following expression.

$Ub = \int_0^s (M^2/2EI)\, ds$

The above expression is similar to the one used in the case of straight beams. However, in this case, the integration needs to be evaluated along the curved arch length. In the above equation, is the length of the centerline of the arch, *sI* is the moment of inertia of the arch cross section, *E* is the Young's modulus of the arch material. The strain energy due to shear is small as compared to the strain energy due to bending and is usually neglected in the analysis. In the case of flat arches, the strain energy due to axial compression can be appreciable and is given by,

$Ua = \int_0^s (N^2/2AE)\, ds$

The total strain energy of the arch is given by,

$U = \int_0^s (M^2/2EI)\, ds + \int_0^s (N^2/2AE)\, ds$

Now, according to the principle of least work $\partial U/\partial H = 0$, where *H* is chosen as the redundant reaction.

$$\partial U/\partial H = \int_0^s (M/EI)\,(\partial M/\partial H)\, ds + \int_0^s (N/AE)\,(\partial N/\partial H)\, ds$$

Solving above equation, the horizontal reaction *H* is evaluated.

Symmetrical two hinged arch

Consider a symmetrical two-hinged arch as shown in Fig 33.2a. Let at crown be the origin of co-ordinate axes. Now, replace hinge at *B* with a roller support. Then we get a simply supported curved beam as shown in Fig 33.2b. Since the curved beam is free to move horizontally, it will do so as shown by dotted lines in Fig 33.2b.

Let *M*0 and *No* be the bending moment and axial force at any cross section of the simply supported curved beam. Since, in the original arch structure, there is no horizontal displacement, now apply a horizontal force *H* as shown in Fig. 33.2c.

The horizontal force *H* should be of such magnitude, that the displacement at B must vanish.

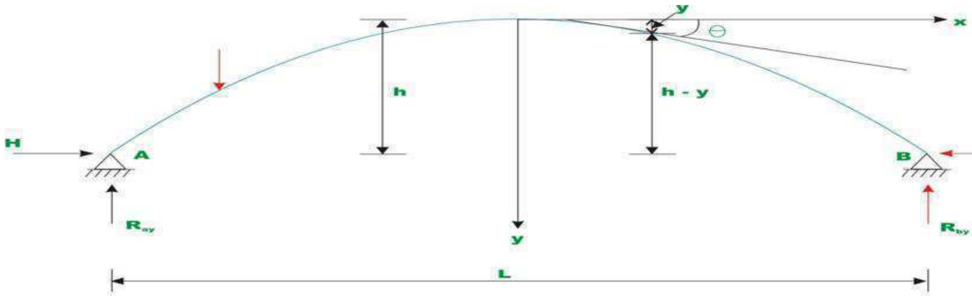


Fig. 33.2a

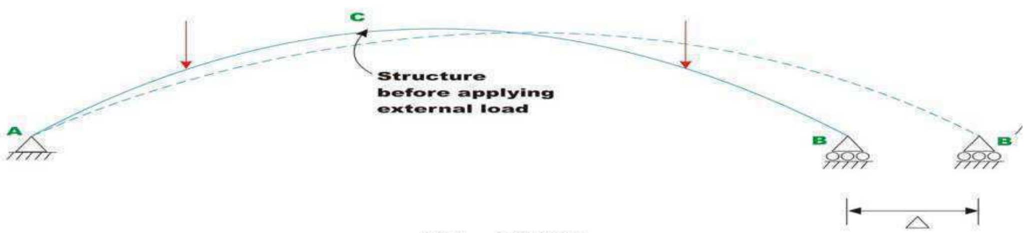


Fig. 33.2b.

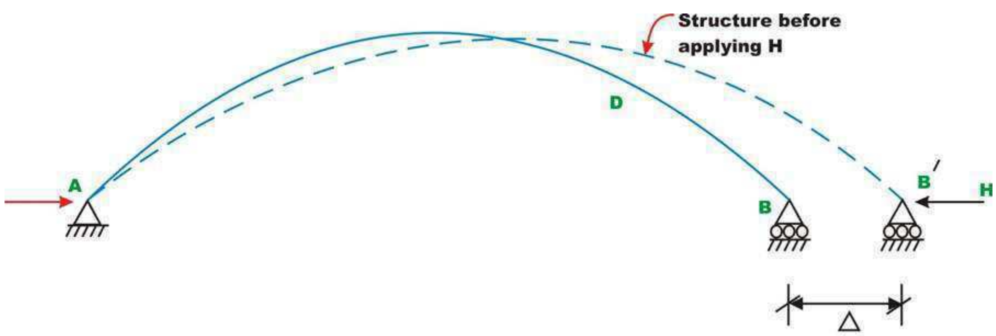


Fig. 33.2c.

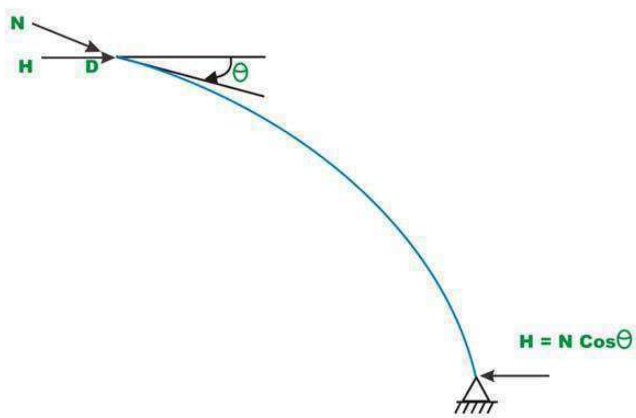


Fig. 33.2d.

From Fig. 33.2b and Fig 33.2c, the bending moment at any cross section of the arch (say), may be written as

M = Mo – H(h – y)

The axial compressive force at any cross section (say) may be written as

N = No + HCos θ

Where θ is the angle made by the tangent at with horizontal (vide Fig 33.2d). ***D***

Substituting the value of *M* and in the equation (33.4),

$$\partial U/\partial H = 0 = -\int_0^s (Mo - H(h-y)(h-y)/EI)\,ds + \int_0^s (No + H \cos\theta \cos\theta / AE)\,ds$$

Solving for *H*, yields,

$$H = \left[\int_0^s (Mo/EI)\, y'ds\right] / \left[\int_0^s (y'^2/AE)\, ds\right]$$

For an arch with uniform cross section *EI* is constant and hence,

$$H = \left[\int_0^s (Mo)\, y'ds\right] / \left[\int_0^s (y'^2)\, ds\right]$$

In the above equation, Mo is the bending moment at any cross section of the arch when one of the hinges is replaced by a roller support. Y' is the height of the arch as shown in the figure. If the moment of inertia of the arch rib is not constant, then equation (33.10) must be used to calculate the horizontal reaction *H*.

Temperature effect

Consider an unloaded two-hinged arch of span *L*. When the arch undergoes a uniform temperature change of *T °C,* then its span would increase by *TLα* if it were allowed to expand freely (vide Fig 33.3a). α is the co-efficient of thermal expansion of the arch material. Since the arch is restrained from the horizontal movement, a horizontal force is induced at the support as the temperature is increased.

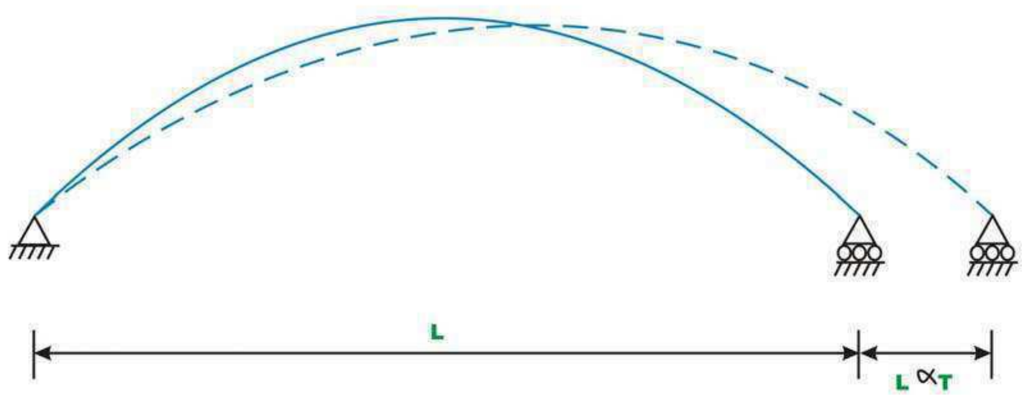


Fig. 33.3a

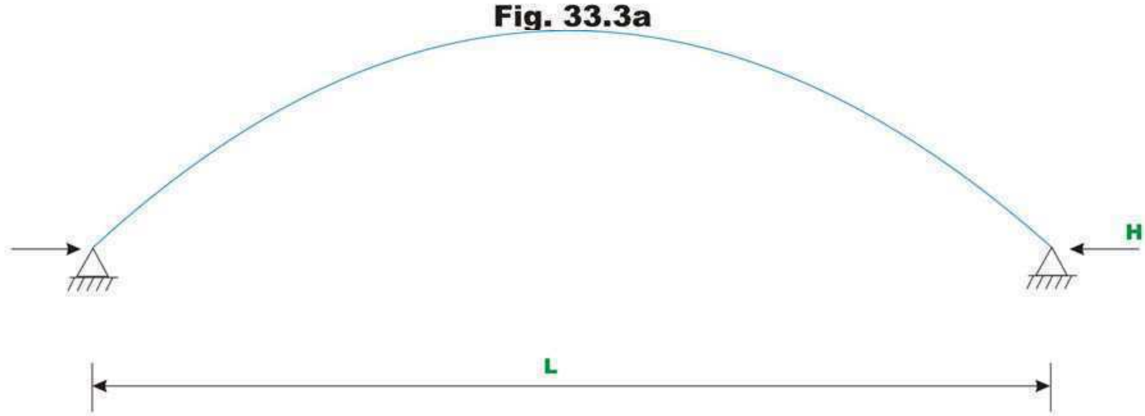


Fig. 33.3b.

Now applying the Castigliano's first theorem,
$\partial U/\partial H = TL\alpha$

Solving for *H*,
$H = [TL\alpha] / [\int_0^s (y'^2/EI)\, ds]$

Example 33.1

A semicircular two hinged arch of constant cross section is subjected to a concentrated load as shown in Fig 33.4a. Calculate reactions of the arch and draw bending moment diagram.

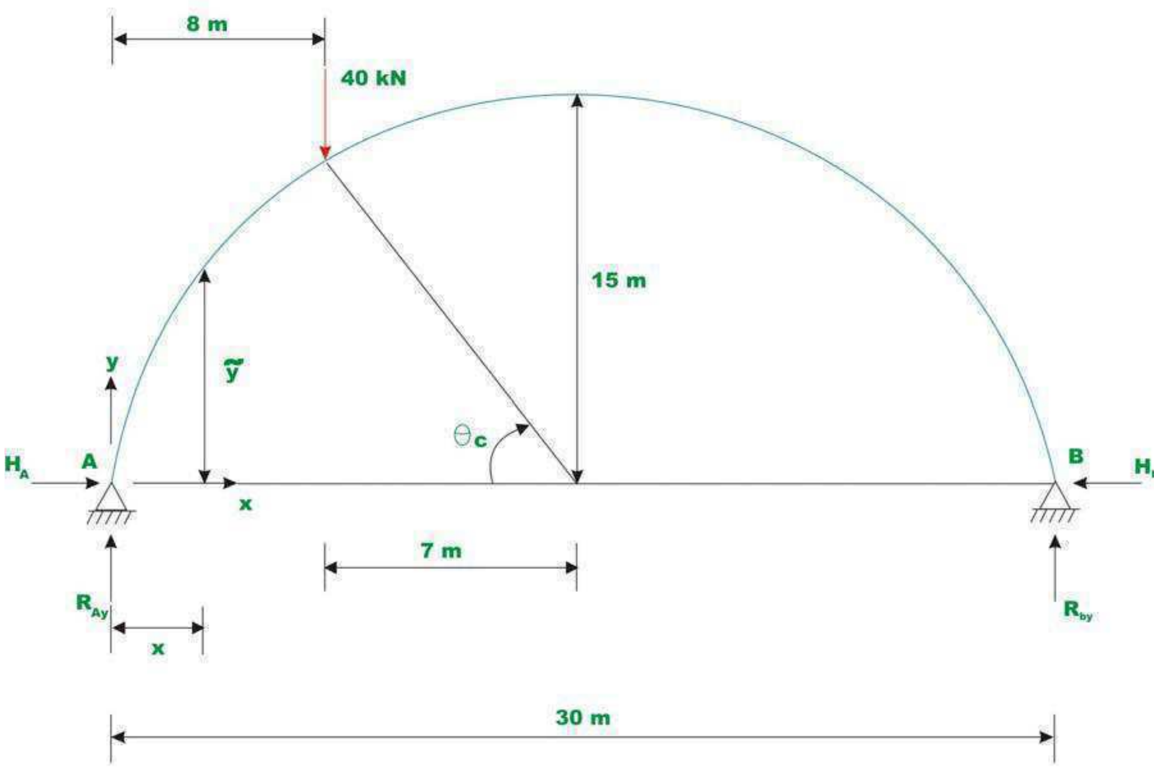


Fig. 33.4a.

Solution:

Taking moment of all forces about hinge *B* leads to,
Va = 29.33 kN.
Vb = 10.67 kN.

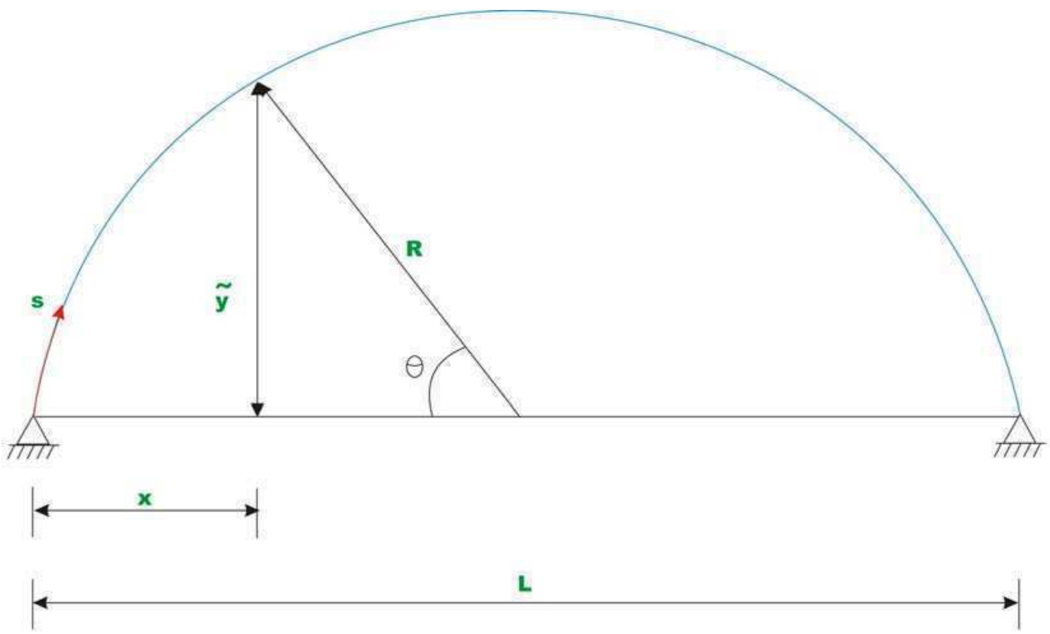


Fig. 33.4b.

From Fig. 33.4b,
Y' = R Sin θ
X = R (1-Cos θ)
Ds = R d θ
Tan θc = 13.267/7
Θc = 62.10°

Now, the horizontal reaction *H* may be calculated by the following expression,
H = [$\int_0^s$(Mo) y'ds] /[$\int_0^s$(y'^{2}) ds]
Now Mo is the bending moment at any cross section of the arch when one of the hinges is replaced by a roller support is given by,

Mo = Va*R (1 - Cos θ) forθc lying between 0 to θ
Mo = Va*R (1 - Cos θ) – 40 (x – 8) for θc lying between θ to π
Integrating and solving, the horizontal thrust at the support is,
H = 19.90 kN.

Bending moment diagram
Bending moment *M*at any cross section of the arch is given by,
M = Mo – Hy' the bending moment diagram is shown in Fig. 33.4c.

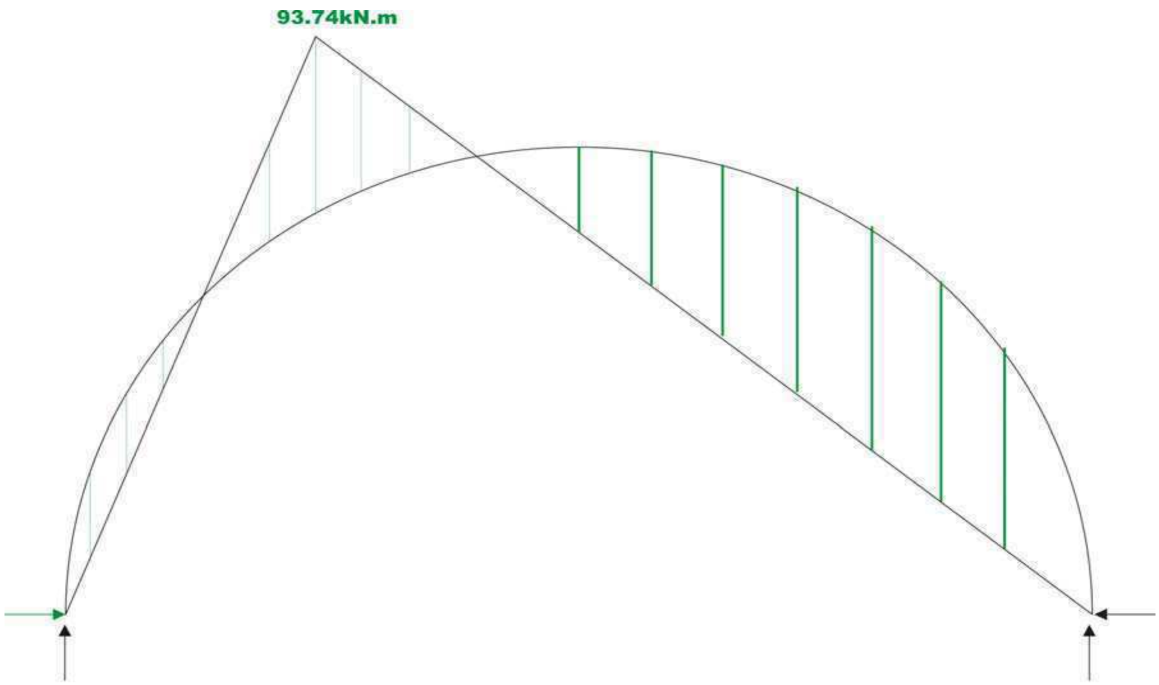


Fig. 33.4c Bending moment diagram

Summary

Two-hinged arch is the statically indeterminate structure to degree one. Usually, the horizontal reaction is treated as the redundant and is evaluated by the method of least work. Towards this end, the strain energy stored in the two-hinged arch during deformation is given. The reactions developed due to thermal loadings are discussed. Finally, a few numerical examples are solved to illustrate the procedure.

Cables

Introduction

Cables and arches are closely related to each other and hence they are grouped in this course in the same module. For long span structures (for e.g. in case bridges) engineers commonly use cable or arch construction due to their efficiency. In the first lesson of this module, cables subjected to uniform and concentrated loads are discussed.

In the second lesson, arches in general and three hinged arches in particular along with illustrative examples are explained. In the last two lessons of this module, two hinged arch and hinge less arches are considered.

Structure may be classified into rigid and deformable structures depending on change in geometry of the structure while supporting the load. Rigid structures support externally applied loads without appreciable change in their shape (geometry). Beams trusses and frames are examples of rigid structures. Unlike rigid structures, deformable structures undergo changes in their shape according to externally applied loads. However, it should be noted that deformations are still small.

Cables and fabric structures are deformable structures. Cables are mainly used to support suspension roofs, bridges and cable car system. They are also used in electrical transmission lines and for structures supporting radio antennas. In the following sections, cables subjected to concentrated load and cables subjected to uniform loads are considered.

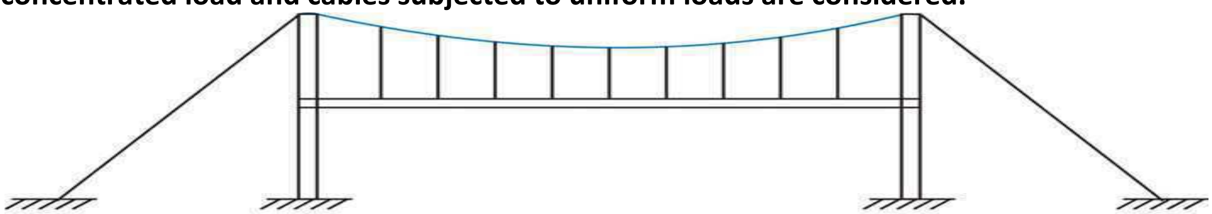


Fig. 31.1 Deformable structure.

Fig 31.2a Unloaded cable
(when dead load is neglected)

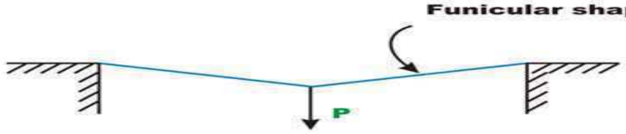


Figure 31.2b Cable in tension.

The shape assumed by a rope or a chain (with no stiffness) under the action of external loads when hung from two supports is known as a funicular shape. Cable is a funicular structure. It is easy to visualize that a cable hung from two supports subjected to external load must be in tension (vide Fig. 31.2a and 31.2b). Now let us modify our definition of cable. A cable may be defined as the structure in pure tension having the funicular shape of the load.

Cable subjected to Concentrated Loads

As stated earlier, the cables are considered to be perfectly flexible (no flexural stiffness) and inextensible. As they are flexible they do not resist shear force and bending moment. It is subjected to axial tension only and it is always acting tangential to the cable at any point along the length. If the weight of the cable is negligible as compared with the externally applied loads then its self-weight is neglected in the analysis. In the present analysis self-weight is not considered.

Cable subjected to uniform load.

Cables are used to support the dead weight and live loads of the bridge decks having long spans.

The bridge decks are suspended from the cable using the hangers. The stiffened deck prevents the supporting cable from changing its shape by distributing the live load moving over it, for a longer length of cable. In such cases cable is assumed to be uniformly loaded.

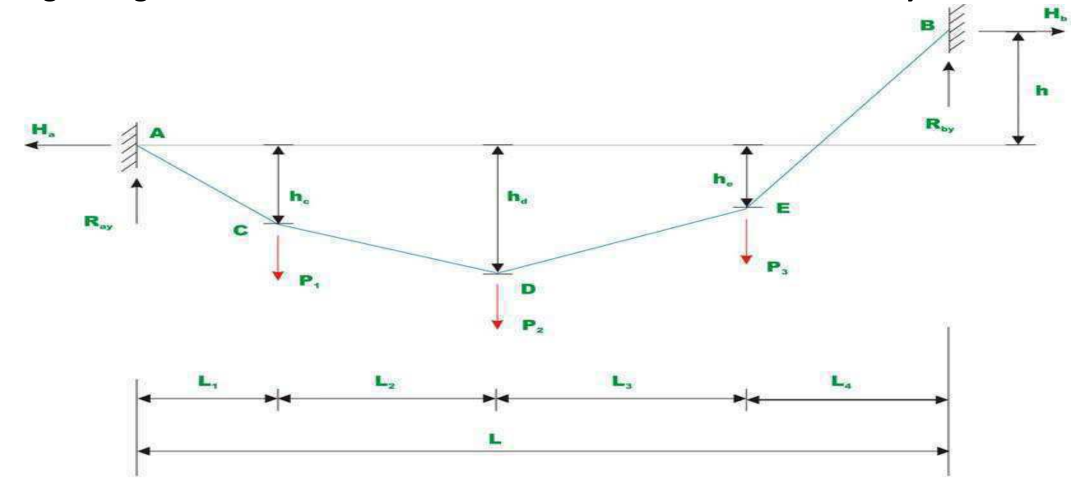
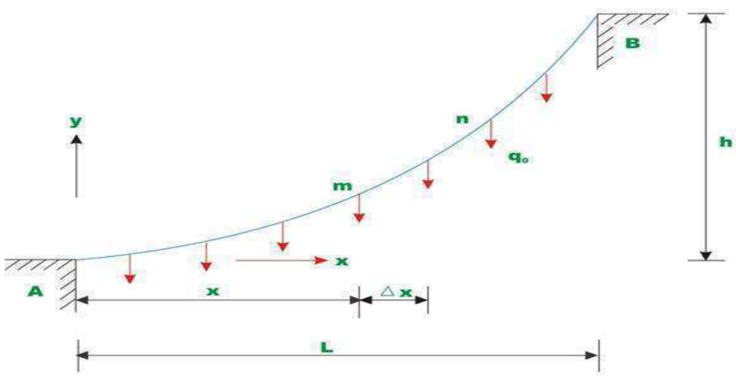


Fig. 31.3a Cable subjected to concentrated load.



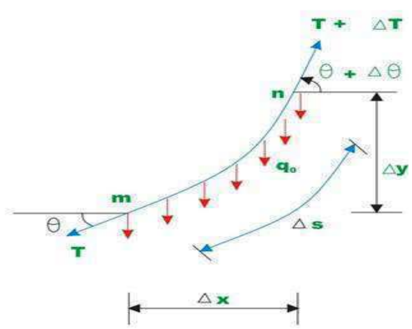


Fig. 31.3b Cable subjected to uniformly distributed load. **Fig. 31.3c Free-body diagram**

Equation of cable $y = q_o x^2 (2H)$

Equation represents a parabola. Now the tension in the cable may be evaluated as

$T = \sqrt{[\, q_o x^2 + H^2]}$

$T = T$ max when $x = L$,

$T \max = \sqrt{[\, q_{oL}^2 + H^2]}$

Due to uniformly distributed load, the cable takes a parabolic shape. However due to its own dead weight it takes a shape of a centenary. However dead weight of the cable is neglected in the present analysis.

Summary

In this lesson, the cable is defined as the structure in pure tension having the funicular shape of the load. The procedures to analyze cables carrying concentrated load and uniformly distributed loads are developed. A few numerical examples are solved to show the application of these methods to actual problem

